Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Chapter 1:

Lecture 3: Total Probability, Bayes' Theorem and Independent Events

Department of Communication & Electronics Engineering

Instructor

Dr. Qadri Hamarsheh

Email: qhamarsheh@philadelphia.edu.jo
Website: http://www.philadelphia.edu.jo/academics/qhamarsheh

Probability

1) Introduction

2) Set Definitions

3) Set Operations

4) Probability Introduced Through Sets and Relative Frequency

5) Joint and Conditional Probability

6) Total Probability and Bayes' Theorem

7) Independent Events

8) Combined Experiments

9) Bernoulli Trials

6)Total Probability and Bayes' Theorem

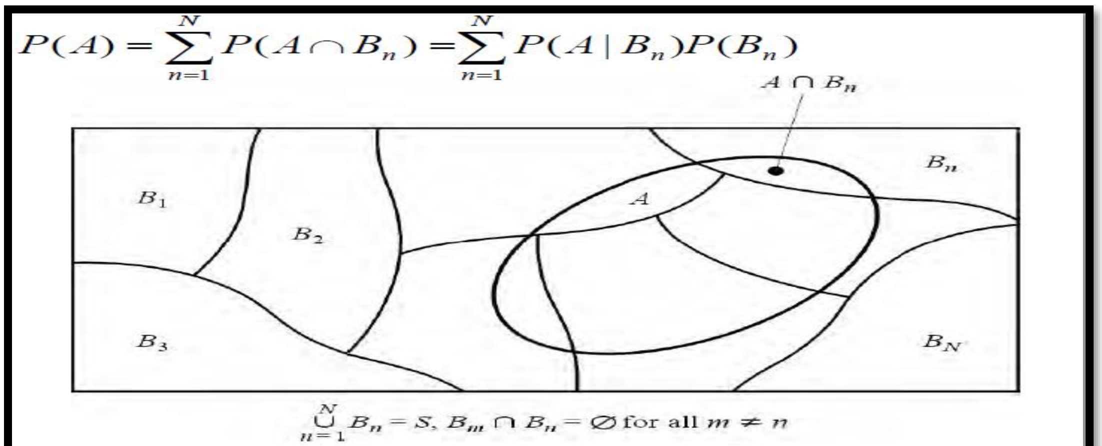
✓ If the events $B_1, B_2, \ldots B_N$ constitute a partition of the sample space **S** such that

1) $\bigcup_{n=1}^{N} B_n = S$ **Collectively Exhaustive**
2) $B_i \cap B_j = \emptyset$ **mutually exclusive**

✓ The **probability** $P(A)$ of any event defined on a sample space S can be expressed in terms of conditional probabilities:

$$P(A) = \sum_{n=1}^{N} P(A|B_n) * P(B_n)$$

Which is known as the **total probability** of event **A**



Prove:

Since $A \cap S = A \Rightarrow$

$$A \cap S = A \cap \left(\bigcup_{n=1}^{N} B_n\right) = \bigcup_{n=1}^{N} (A \cap B_n)$$

The events $(A \cap B_n)$ are mutually exclusive, then

$$P(A) = P(A \cap S) = P\left[\bigcup_{n=1}^{N} (A \cap B_n)\right] = \sum_{n=1}^{N} P((A \cap B_n))$$

- ✓ **Bayes' Theorem**:
    - Bayes' rule is one of the most important rules in probability theory.
    - Bayes' theorem is often referred to as a theorem on the probability of causes.
    - From the conditional probability (If $P(A) \neq 0$ and $P(B) \neq 0$):

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

and

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

- Bayes' theorem is obtained by **equating these two expressions**:

$$P(A \cap B) = P(A|B) * P(B) = P(B|A) * P(A) \qquad i$$

$$P(A|B) * P(B) = P(B|A) * P(A)$$

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)} \qquad ii(Bayes' \text{ theorem})$$

- Sample space and intersection of events: **Generalized Bayes' theorem**
  - In the case of three events **A_1, A_2, A_3** are mutually exclusive and collectively exhaustive (probabilities of all events = 1 and construct the sample space)
  - B has some common in **A_1, A_2,** and **A_3**

$$B = (A_1 \cap B) U (A_2 \cap B) \cup (A_3 \cap B)$$

From equation *(i)*

$$P(B) = P(B|A_1) * P(A_1) + P(B|A_2) * P(A_2) + P(B|A_3) * P(A_3)$$

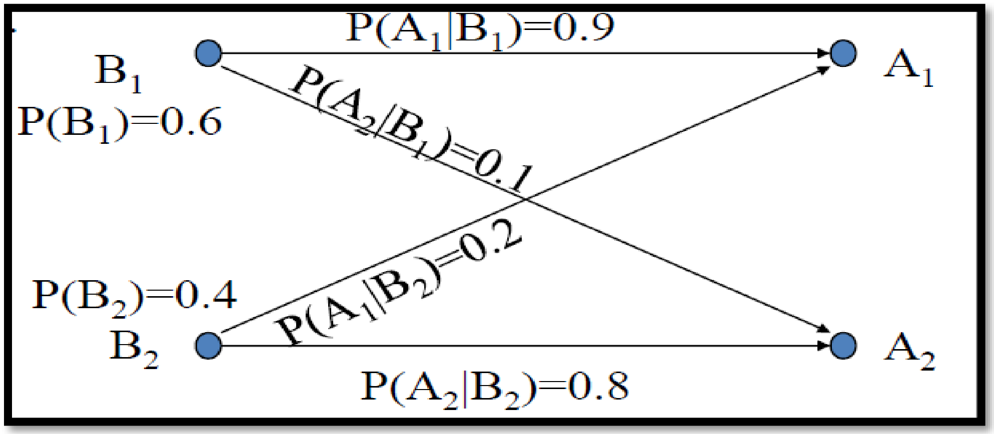
$$P(B) = \sum_{i=1}^{n} P(B|A_i) * P(A_i) \qquad iii$$

✓ Putting all together (substitute ***iii*** in ***ii***) we get

$$P(A|B) = \frac{P(B|A) * P(A)}{\sum_{i=1}^{n} P(B|A_i) * P(A_i)}$$

Generalized form of the Bayes' Theorem

✓ **Example:** An elementary binary communication system consists of a transmitter that sends one of two possible symbols (a **1** or a **0**) over a channel to a receiver. The channel occasionally causes errors to occur so that a 1 shows up at the receiver as a 0, and vice versa.



Denote by $B_i; i = 1, 2$, as the events the symbol before the channel and $A_i; i = 1, 2$, as the events the symbol after the channel.

- The **probabilities of receiving symbol** " $A1$" and " $A2$" are:

$$P(A_1) = P(A_1|B_1)\,P(B_1) + P(A_1|B_2)\,P(B_2) = (0.9)(0.6)+(0.2)(0.4)=0.62$$
$$P(A_2) = P(A_2|B_1)\,P(B_1) + P(A_2|B_2)\,P(B_2) = (0.1)(0.6)+(0.8)(0.4)=0.38$$

- The probability that $B1$ is sent if **A1** is received (using Bayes' theorem):

$$P(B_1 \mid A_1) = \frac{P(A_1 \mid B_1).P(B_1)}{P(A_1)} = \frac{(0.9)(0.6)}{0.62} = 0.87$$

- Similarly for:

$$P(\,B2|\,A1) = \;0.13\,,$$
$$P(\,B2|\,A2) = 0.84$$
$$P(\,B1|\,A2) = 0.16$$

- The **total probability of error** for the system:

$$P_e \;=\; P(\,A_1|\,B_2)\,P(\,B_2)\;+\;P(\,A_2|\,B_1)P(B_1)) \;= (0.2)(0.4) + (0.1)(0.6) = 0.14$$

✓ **Example:** In a certain assembly plant, three machines, B_1, B_2, and B_3, make **30%, 45%,** and **25%** respectively of the products. It is known from past experience that **2%, 3%** and **2%** of the products made by each machine, respectively, are defective. Now suppose that a finished product is randomly selected.

a) What is the probability that it is defective?

- **Solution**:

  Events: **A** the product is defective

  B_1 the product is made by machine B_1

  B_2 the product is made by machine B_2

  B_3 the product is made by machine B_3

  Using total probability theorem

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3) =$$
$$0.3 * 0.02 + 0.45 * 0.03 + 0.25 * 0.02 =$$
$$0.006 + 0.0135 + 0.005 = 0.0245$$

b) If a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

- **Solution**: Using Bayes' rule

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$P(B_3|A) = \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{0.005}{0.0245} = \frac{10}{49}$$

- ✓**Example:** A box contains **6 green balls**, **4 black balls**, and **10 yellow balls**. All balls are equally likely (probable) to be drawn. What is the probability of drawing **two green balls** from the box if the first drawn ball is not replaced?

$$P(G \cap G) = P(G|G)\,P(G) = (5/19)(6/20) = 0.0789$$

- ✓In Bayes' theorem $P(B_n)$ are usually referred to as a **Priori probabilities**.
- ✓The $P(A|B_n)$ are numbers typically known prior to conducting the experiment.
- ✓The $P(B_n|A)$ are called a **Posteriori probabilities**

7)Independent Events

- ✓The two nonzero probabilities events A and B are called **statistically independent** if the probability of occurrence of one event is not affected by the occurrence of the other event.
- ✓If events A and B are **statistically independent** then **they can both occur**.
- ✓**Mathematically**:
  - **First approach** for **testing** the Independent Events

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

Conditions **B** doesn't affect in **A.**

✓ **Example:** From HBO example from previous lecture

$$P(WW|F) = \frac{0.05}{0.54} = 0.093$$

$$P(WW) = 0.25$$

Therefore the events are **NOT** independent as $0.093 \neq 0.25$

- **Second approach** for **testing** the Independent Events:

$$P(A \cap B) = P(A) \,.\, P(B) \quad \textbf{(Product rule)}$$

✓ **Example:** From HBO example from previous lecture

$$P(WW \cap F) = 0.05$$

And

$$P(WW) * P(F) = 0.25 * 0.54 = 0.14$$

Therefore **NOT** independent as $0.05 \neq 0.14$

✓ **Example:** rolling a Dice and flipping a coin are independent events, so the probability of getting 2 and a Heads is

$$P(2 \cap H) = P(2) * P(H) = \frac{1}{6} * \frac{1}{2}$$

✓ **Example:** In an experiment, one card is selected from an ordinary 52-card deck. Define events:

A = {select a king}
B = {select a jack or queen}
C = {select a heart}

We find:

$$P(A) = 4/52, P(B) = 8/52, P(C) = 13/52$$
$$P(A \cap B) = 0 \neq P(A)P(B) = (4/52)(8/52)$$
$$P(A \cap C) = 1/52 = P(A)P(C) = (4/52)(13/52)$$
$$P(B \cap C) = 2/52 = P(B)P(C) = (8/52)(13/52)$$

Thus,

- A and B are **not independent**
- A and C are **independent**
- B and C are **independent**
- A and B are **mutually exclusive**

✓ **Multiple Events**: In the case of three events A_1, A_2, A_3 are independent if they are pairwise independent:

$$P(A_j \cap A_k) = P(A_j) * P(A_k), \quad j \neq k = 1, 2, 3, \ldots$$

And also independent as a triple

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) * P(A_2) * P(A_3)$$

- ✓ **Example:** Find the probability of drawing an ace on the first, second, third, and fourth cards from an ordinary 52-card deck. Assume that:
  a) Each is **replaced** after drawing

$$P(A1 \cap A2 \cap A3 \cap A4) = P(A1)\,P(A2)\,P(A3)\,P(A4)$$
$$= (4/52)(4/52)(4/52)(4/52)$$

Events $A1, A2, A3$, and $A4$ are independent

  b) Each is **not replaced** after drawing

$$P(A1 \cap A2 \cap A3 \cap A4) = (4/52)(3/51)(2/50)(1/49)$$

Events $A1, A2, A3$ and $A4$ are not independent

- ✓ It the two events have nonzero probabilities then by comparing mutually exclusive events $P(A \cap B) = 0$ and statistically independent events $P(A \cap B) = P(A) * P(B)$, we can easily establish that two events cannot be both mutually exclusive and statistically independent.